The Universal Portfolio

Alex Plotnick

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NOTES: Introduction

Intuitively, one might think that an optimal stock-market investment strategy might require either the existence and knowledge of some underlying distribution for the market, or knowledge of the future. The universal portfolio is an investment strategy that requires neither, but that nonetheless performs asymptotically as well as one that does.
References


NOTES: References

[Cov91] lays out the basic algorithm.

[CO96] extends this to side-information and improves the performance bounds of the algorithm.
Basic Definitions

In a market of \( m \) stocks, let \( x = (x_1, x_2, \ldots, x_m)^t \) be a stock vector, where each \( x_j \) is the price relative of the \( j^{\text{th}} \) stock.

Investments are specified by a portfolio vector

\[
    b = (b_1, b_2, \ldots, b_m)^t \in B,
\]

where

\[
    B = \left\{ b \in \mathbb{R}^m : \sum_{j=1}^{m} b_j = 1, b_j \geq 0 \right\}.
\]

If \( b \) remains constant throughout the day, we call \( b \) a constant rebalanced portfolio (CRP).
NOTES: Basic Definitions

The *price relative* of a stock is the ratio of its final price to its initial price over some investment period (e.g., one day). So it is a unitless measure of change in a stock’s price.

A portfolio \( b \) is a division of wealth amongst \( m \) stocks, so that the \( j^{th} \) entry \( b_j \) of \( b \) is the proportion of wealth invested in the \( j^{th} \) stock. Thus \( b \) is an element of the \((m - 1)\)-dimensional simplex \( \mathcal{B} \). All of the portfolios discussed in this talk will be CRPs. It is well known that if the price relatives are i.i.d., the optimal growth rate of wealth is achieved by a CRP, and that this growth rate is exponential.
Basic Definitions (continued)

An investment using a portfolio $b$ increases one’s wealth in a day by a factor of

$$S = b^t x = \sum_{j=1}^{m} b_j x_j.$$ 

Over $n$ days, using portfolios $b_i$ for $i = 1, \ldots, n$, wealth increases by a factor of

$$S_n = \prod_{i=1}^{n} b_i^t x_i = \prod_{i=1}^{n} \sum_{i=1}^{m} b_{ij} x_{ij},$$

if market performance is $x^n = (x_1, x_2, \ldots, x_n)$.

A sequence of such portfolio choices $b_i$ is called an investment strategy.
NOTES: Basic Definitions (continued)

The per-day increase in wealth is the dot-product of the portfolio vector with the stock vector.

$S_n$ is simply the per-day wealth increase multiplied over $n$ days.
Best CRP

Let the \textit{target wealth}

\[ S_n^* = \max_b S_n(b) \]

be the maximum achievable wealth for the given stock sequence, maximized over all possible CRPs. Our goal is to achieve \( S_n^* \).
NOTES: Best CRP

With foreknowledge, $S^*_n$ is achievable. The question is, can we achieve this goal without knowledge of the future?
Let $b = e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ denote the “buy-and-hold” strategy for the $j^{th}$ stock. Then we have:

\begin{align*}
S_n^* & \geq \max_{j=1,\ldots,m} S_n(e_j) \quad (1) \\
S_n^* & \geq \left( \prod_{j=1}^{m} S_n(e_j) \right)^{1/m} \quad (2) \\
S_n^* & \geq \sum_{j=1}^{m} \alpha_j S_n(e_j) \quad (3)
\end{align*}

Also note that $S_n^*(x_1, x_2, \ldots, x_n)$ is invariant under permutations of the sequence $x_1, x_2, \ldots, x_n$. 
These are some properties of the best CRP. Equation 1 says that our target exceeds the best stock. Eq. 2 says that it exceeds the value line index, and eq. 3 says that it exceeds any arithmetic mean (e.g., DJIA, S&P 500).

Permutation invariance means that, e.g., a crash will do no more harm than if the bad days had been sprinkled out among the good.
A Simple Example

Consider the following sequence of stock market vectors:

\[ x_1, x_2, \ldots = (1, 2)^t, \left(1, \frac{1}{2}\right)^t, (1, 2)^t, \left(1, \frac{1}{2}\right)^t, \ldots \]

Buy-and-hold of the first stock results in \( S_n = 1 \); buy-and-hold of the second stock results in \( S_n = 1 \) when \( n \) is even.

A CRP of \( b^* = \left(\frac{1}{2}, \frac{1}{2}\right) \) yields wealth \( S^*_n = \left(\frac{3}{\sqrt{8}}\right)^n \approx 1.06^n \) when \( n \) is even; i.e., wealth goes exponentially to infinity.
NOTES: A Simple Example

We have two stocks, one of which remains constant (e.g., a risk-free asset (cash)), the other of which jumps up and down by a factor of two every day.

The derivation of $S^*_n$ is a simple maximization by taking the derivative and setting it equal to zero [CO96, page 24].
The Universal Portfolio

The universal portfolio strategy $\hat{b}_k$ is defined as:

$$\hat{b}_1 = \left( \frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m} \right)^t,$$

$$\hat{b}_{k+1} = \frac{\int_{B} b S_k(b) \, db}{\int_{B} S_k(b) \, db},$$

where

$$S_k(b) = \prod_{i=1}^{k} b^t x_i.$$
NOTES: The Universal Portfolio

We begin with our initial investment evenly divided among the \( m \) stocks. On each successive day, we calculate a new portfolio by taking the performance-weighted average of all possible portfolios.

It is important and interesting to note that the definition of the universal portfolio makes no assumptions about any underlying distribution of the stock vectors. This is in sharp contrast to most other investment strategies.
Wealth Generated by the Universal Portfolio

The wealth generated by the universal portfolio on day \( n \) is:

\[
\hat{S}_n = \prod_{k=1}^{n} \hat{b}_k^{t_{x_k}}.
\]

**Theorem 1**

\[
\hat{S}_n = \int S_n(b) \, db / \int db,
\]

where

\[
S_n(b) = \prod_{i=1}^{n} b_{x_i}^t.
\]
NOTES: Wealth Generated by the Universal Portfolio

Wealth $\hat{S}_n$ resulting from the universal portfolio is the average of $S_n(b)$ over the simplex. The proof is a straightforward manipulation of the definition [Cov91, page 6].

One can also prove that the wealth $\hat{S}_n$ has some of the properties of the target wealth $S^*_n$: it exceeds the value line index, and, although $\hat{b}_k$ depends on the order of the stock vector sequence, the resulting wealth does not. Thus, stock market crashes will do no more harm to the resultant wealth than if the bad days were sprinkled throughout the good.

The reason the portfolio works is that the growth rate of CRPs is exponential, and the average of exponentials has the same asymptotic growth rate as the maximum.
General Universal Portfolio

The $\mu$-weighted universal portfolio at time $i$ is given by:

$$\hat{b}_i = \hat{b}_i(x^{i-1}) = \frac{\int_B b S_{i-1}(b, x^{i-1}) d\mu(b)}{\int_B S_{i-1}(b, x^{i-1}) d\mu(b)},$$

with

$$\int_B d\mu(b) = 1,$$

and, as before,

$$S_i(b, x^i) = S_i(b) = \prod_{j=1}^{i} b^t x_j, \text{ and } S_0(b, x^0) = 1.$$ 

Note that if $\mu$ is symmetric, then

$$\hat{b}_1 = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)^t.$$
NOTES: General Universal Portfolio

The measure $\mu$ can be thought of as putting some probability mass distribution on each face of the simplex $B$. This definition is equivalent to the previous definition for $\mu$ equal to the uniform distribution.
Dirichlet($1/2, 1/2$) Distribution
NOTES: Dirichlet(1/2,1/2) Distribution

The authors of [CO96] focus on two distributions: the uniform (Dirichlet(1,\ldots,1)) and the Dirichlet(1/2,\ldots,1/2) distributions.
Performance of the Universal Portfolio

Let $m$ be the number of assets in our portfolio, and $x^n$ be the sequence of stock vectors up to day $n$.

Theorem 2  For $\mu$ equal to the uniform distribution,

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} \leq \binom{n + m - 1}{m - 1} \leq (n + 1)^{m-1}.$$

Theorem 3  For $\mu$ equal to the Dirichlet$(1/2, \ldots, 1/2)$ distribution,

$$\frac{S_n^*(x^n)}{\hat{S}_n(x^n)} \leq \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n + \frac{m}{2})}{\Gamma\left(\frac{m}{2}\right)\Gamma(n + \frac{1}{2})} \leq 2(n + 1)^{\frac{m-1}{2}}.$$

That is, the wealth generated by the universal portfolio for these two distributions is within a polynomial factor of the wealth generated by the best CRP.
NOTES: Performance of the Universal Portfolio

The important thing to note here is that the universal portfolio is achieving this wealth on the fly, whereas the best CRP depends on hindsight.
Whence Compression?

The universal portfolio is *universal* in the same sense that Lempel-Ziv coding is universal: as $n \to \infty$, they achieve asymptotically optimal results (e.g., maximum wealth for the portfolio, compression $\sim$ entropy for LZ).
NOTES: Whence Compression?

The authors of [CO96] make this more precise, but the connection is made by examining certain bounds in the proof.